Konvektionsgleichungen

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Navier - Stokes - Gl.

$$\rho \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_i$$

Boussinesq - Näherung : - nur kleine Dichtevariationen
- nur im Kraftterm berücksichtigt

$$\rho_0 \frac{Du_i}{Dt} = -\frac{\partial p}{\partial x_i} + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \rho f_i$$

ρ bei Ref. temp. T_0 $\quad$ $\mu \neq f(T)$ $\quad$ $\rho(T)$ variabel

$$\longrightarrow \frac{Du_i}{Dt} = -\frac{1}{\rho_0}\frac{\partial p}{\partial x_i} + \frac{\mu}{\rho_0}\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\rho}{\rho_0} f_i$$

$$\frac{\rho}{\rho_0} f_i = \frac{\rho}{\rho_0}\begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix} = -\frac{\rho}{\rho_0}\frac{\partial z}{\partial x_i} g + \underbrace{\frac{\partial}{\partial x_i}(gz - gz)}_{g\frac{\partial z}{\partial x_i} - \frac{\partial}{\partial x_i}\left(\frac{\rho_0 g z}{\rho_0}\right)}$$

$$= -\frac{\rho - \rho_0}{\rho_0}\frac{\partial z}{\partial x_i} g - \frac{\partial}{\partial x_i}\left(\frac{\rho_0 g z}{\rho_0}\right)$$

Ausdehnungskoeffizient für kleine Variationen von ρ

$$\beta = -\frac{1}{\rho}\left(\frac{\partial \rho}{\partial T}\right)_p \approx -\frac{1}{\rho_0}\frac{\Delta\rho}{\Delta T} \approx \frac{1}{T_0} \longrightarrow -\frac{\Delta\rho}{\rho_0} = \frac{\Delta T}{T_0}$$

Eingesetzt

$$\frac{Du_i}{Dt} = -\frac{\partial}{\partial x_i}\left(\frac{p + \rho_0 g z}{\rho_0}\right) + \nu \frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{T - T_0}{T_0}\frac{\partial z}{\partial x_i} g$$

$$\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}$$

dim. lose Darstellung

$$u_i = u_i^* U_\infty \qquad x_i = x_i^* L_\infty \qquad t = t^* L_\infty / U_\infty \longrightarrow \begin{cases} \dfrac{\partial u_i}{\partial t} = \dfrac{\partial u_i^*}{\partial t^*} \cdot \dfrac{U_\infty^2}{L_\infty} \\[2ex] u_j \dfrac{\partial u_i}{\partial x_j} = u_j^* \dfrac{\partial u_i^*}{\partial x_j^*} \cdot \dfrac{U_\infty^2}{L_\infty} \\[2ex] \dfrac{\partial^2 u_i}{\partial x_j \partial x_j} = \dfrac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} \cdot \dfrac{U_\infty}{L_\infty^2} \end{cases}$$

$$\longrightarrow \frac{U_\infty^2}{L_\infty}\left(\frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*}\right) = -\frac{1}{L_\infty}\frac{\partial}{\partial x_i^*}\left(\frac{p + \rho_0 g z}{\rho_0}\right) + \nu \frac{U_\infty}{L_\infty^2} \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + g \frac{\partial z^*}{\partial x_i^*} \underbrace{\frac{T - T_0}{T_0}}_{T^*}$$

$$\longrightarrow \frac{\partial u_i^*}{\partial t^*} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial}{\partial x_i^*}\underbrace{\left(\frac{p + \rho_0 g z}{\rho_0 U_\infty^2}\right)}_{p^*} + \underbrace{\frac{\nu}{U_\infty L_\infty}}_{Re^{-1}} \frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} + \underbrace{\frac{g L_\infty}{U_\infty^2}}_{Fr^{-2}} \frac{\partial z^*}{\partial x_i^*} T^*$$

Poisson - Gl. aus Divergenz der Impulsgl.

$$\frac{\partial}{\partial x_i^*}\left(\cancel{\frac{\partial u_i^*}{\partial t^*}} + u_j^* \frac{\partial u_i^*}{\partial x_j^*} = -\frac{\partial p^*}{\partial x_i^*} + Re^{-1}\cancel{\frac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*}} + Fr^{-2}\frac{\partial z^*}{\partial x_i^*} T^*\right)$$

$$\rightarrow \frac{\partial u_i^* u_j^*}{\partial x_j^*}$$ mit Boussinesq - Annahme

$$\begin{cases} \dfrac{\partial}{\partial x_i^*}\dfrac{\partial u_i^*}{\partial t^*} = \dfrac{\partial}{\partial t^*}\cancel{\dfrac{\partial u_i^*}{\partial x_i^*}} = 0 & \text{mit Boussinesq - Annahme} \\[2ex] \dfrac{\partial}{\partial x_i^*}\dfrac{\partial^2 u_i^*}{\partial x_j^* \partial x_j^*} = \dfrac{\partial^2}{\partial x_j^* \partial x_j^*}\cancel{\dfrac{\partial u_i^*}{\partial x_i^*}} = 0 & \text{mit Boussinesq - Annahme} \end{cases}$$

$$\longrightarrow \frac{\partial^2 u_i^* u_j^*}{\partial x_i^* \partial x_j^*} = -\frac{\partial^2 p^*}{\partial x_i^* \partial x_i^*} + Fr^{-2}\frac{\partial z^*}{\partial x_i^*}\frac{\partial T^*}{\partial x_i^*}$$

Energiegleichung für inkompressible Fluide ⟶ Boussinesq - Annahme

$$\varrho_0 c_p \frac{DT}{Dt} = \cancel{\underline{\underline{\tau}} : \underline{\underline{\nabla u}}} + \varrho_0 q_v + \lambda \frac{\partial^2 T}{\partial x_j \partial x_j}$$

Dissipation vernachlässigbar

$\hat{=} Q$ Wärmequellen

$$\varrho_0 c_p \frac{DT^*}{Dt^*} \frac{T_0 u_\infty}{L_\infty} = Q + \lambda \frac{\partial^2 T^*}{\partial x_j^* \partial x_j^*} \frac{T_0}{L_\infty^2}$$

$$\longrightarrow \frac{DT^*}{Dt^*} = \underbrace{\frac{\lambda}{\varrho_0 c_p \nu}}_{Pr^{-1}} \underbrace{\frac{\nu}{u_\infty L_\infty}}_{Re^{-1}} \frac{\partial^2 T^*}{\partial x_j^* \partial x_j^*} + \underbrace{\frac{Q L_\infty}{\varrho_0 c_p u_\infty T_0}}_{Q^*}$$